

Triple Linking of Surfaces in 4-Space*

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Abstract

Triple linking numbers were defined for 3-component oriented surface-links in 4-space using signed triple points on projections in 3-space. In this paper we give an algebraic formulation using intersections of homology classes (or cup products on cohomology groups). We prove that spherical links have trivial triple linking numbers and that triple linking numbers are link homology invariants.

1 Introduction

A *surface-link* is a closed surface F embedded in \mathbf{R}^4 locally flatly. In this paper, we always assume that F is *oriented*, that is, each component of F is orientable and given a fixed orientation. For a 3-component surface-link $F = K_1 \cup K_2 \cup K_3$, a linking number was defined in [1] using its projection in \mathbf{R}^3 in a way that is analogous to the linking number in classical knot theory. In that paper, it was introduced as an example of non-triviality of the state-sum invariants of surface-links. In fact, the state-sum invariants in the classical link and surface-link case generalize linking number and Fox's coloring number.

In the current paper, we give several alternative definitions of the triple linking number and some properties. The reader will find that this invariant is a quite natural generalization of the notion of classical linking number in contrast to a statement in Rolfsen [11] page 136: "There is, however no analogous notion of linking number to help us with codimension two link theory, for example, in higher dimensions". We note, however, that Rolfsen himself with Massey [10] and with Fenn [5] generalized classical linking numbers to higher dimensions using degrees of maps.

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Let $F = K_1 \cup K_2 \cup K_3$ be a 3-component surface-link in \mathbf{R}^4 . It is known (see [2] for example) that a projection of F into \mathbf{R}^3 can be assumed to have transverse double curves and isolated branch/triple points. At a triple point, three sheets intersect that have distinct relative heights with respect to the projection direction, and we call them top, middle, and bottom sheets, accordingly. If the orientation normals to the top, middle, bottom sheets at a triple point τ matches with this order the fixed orientation of \mathbf{R}^3 , then the sign of τ is positive and $\varepsilon(\tau) = 1$. Otherwise the sign is negative and $\varepsilon(\tau) = -1$. (See [1, 2].) It is also known that any closed oriented embedded surface F in \mathbf{R}^4 bounds an oriented compact 3-manifold M embedded in \mathbf{R}^4 , called a *Seifert hypersurface* of F , such that $\partial M = F$.

We give six methods for defining an integer (triple linking number) as follows.

- (1) Consider a surface diagram of $K_1 \cup K_2 \cup K_3$ in \mathbf{R}^3 . A triple point is *of type* (i, j, k) if the top sheet comes from K_i , the middle comes from K_j , and the bottom comes from K_k . The sum of the signs of all the triple points of type $(1, 2, 3)$ is denoted by $\text{Tlk}_1(K_1, K_2, K_3)$. This is the definition given in [1].
- (2) Let M_i be a Seifert hypersurface for K_i ($i = 1, 3$). Assume that $M_i \cap K_2$ is a 1-manifold in K_2 and that $M_1 \cap K_2$ and $M_3 \cap K_2$ intersect transversely. Count the intersections between them algebraically and denote the sum by $\text{Tlk}_2(K_1, K_2, K_3)$.
- (3) Consider a Seifert hypersurface M_1 for K_1 . Assume that $M_1 \cap K_2$ is a 1-manifold, which is disjoint from K_3 . The linking number $\text{Link}(M_1 \cap K_2, K_3)$ is denoted by $\text{Tlk}_3(K_1, K_2, K_3)$.
- (4) Let M_i be a Seifert hypersurface for K_i ($i = 1, 3$) such that $M_1 \cap M_3$ is a 2-manifold which intersects K_2 transversely. Count the intersections between them algebraically and denote the sum by $\text{Tlk}_4(K_1, K_2, K_3)$.
- (5) Let M_i be a Seifert hypersurface for K_i ($i = 1, 2, 3$) and let N_2 be a regular neighbourhood of K_2 in \mathbf{R}^4 . We may assume that $M_i \cap \partial N_2$ is a 2-manifold in ∂N_2 and that $M_1 \cap \partial N_2$, $M_2 \cap \partial N_2$ and $M_3 \cap \partial N_2$ intersect transversely in a finite number of points. Count the intersections algebraically and denote the sum by $\text{Tlk}_5(K_1, K_2, K_3)$.
- (6) Let $f : F_1 \cup F_2 \cup F_3 \rightarrow \mathbf{R}^4$ denote an embedding of the disjoint union of oriented surfaces F_i representing $F = K_1 \cup K_2 \cup K_3$. Define a map $L : F_1 \times F_2 \times F_3 \rightarrow S^3 \times S^3$ by

$$L(x_1, x_2, x_3) = \left(\frac{f(x_1) - f(x_2)}{\|f(x_1) - f(x_2)\|}, \frac{f(x_2) - f(x_3)}{\|f(x_2) - f(x_3)\|} \right)$$

for $x_1 \in F_1$, $x_2 \in F_2$ and $x_3 \in F_3$, and denote the degree of L by $\text{Tlk}_6(K_1, K_2, K_3)$.

Theorem 1.1 $\text{Tlk}_i(K_1, K_2, K_3) = \pm \text{Tlk}_j(K_1, K_2, K_3)$ for any $i, j = 1, \dots, 6$.

Remark. In general, the *triple linking number* $\text{Tlk}(K_i, K_j, K_k)$ for $i \neq j \neq k$ is defined to be the sum of the signs of all the triple points of type (i, j, k) on a surface diagram of F ;

$$\text{Tlk}(K_i, K_j, K_k) = \sum_{\tau: \text{type } (i, j, k)} \varepsilon(\tau).$$

It is proved in [1] that this number is an invariant of the surface-link F (independent of a diagram in \mathbf{R}^3) by use of Roseman moves (Reidemeister moves for surface-link diagrams) [12], and that this invariant vanishes in the case that $i = k$; that is, $\text{Tlk}(K_i, K_j, K_i) = 0$ for $i \neq j$. Hence throughout this paper, we always assume that i, j, k are all distinct whenever we refer to $\text{Tlk}(K_i, K_j, K_k) = \text{Tlk}_1(K_i, K_j, K_k)$.

We prove the following properties of triple linking by using the above interpretations.

Theorem 1.2 ([1]) (i) $\text{Tlk}(K_1, K_2, K_3) = -\text{Tlk}(K_3, K_2, K_1)$.

(ii) $\text{Tlk}(K_1, K_2, K_3) + \text{Tlk}(K_2, K_3, K_1) + \text{Tlk}(K_3, K_1, K_2) = 0$.

Theorem 1.3 (i) *If K_2 is homeomorphic to a 2-sphere, then $\text{Tlk}(K_1, K_2, K_3) = 0$.*

(ii) *If both of K_1 and K_3 are homeomorphic to a 2-sphere, then $\text{Tlk}(K_1, K_2, K_3) = 0$.*

In [14] the asymmetric linking number $\text{Alk}(K, K')$ for a two component oriented surface-link $F = K \cup K'$ was defined to be the non-negative generator of the image of $H_1(K) \rightarrow H_1(S^4 \setminus K') \cong \mathbf{Z}$.

Theorem 1.4 *If $\text{Alk}(K_2, K_3) = 0$, then $\text{Tlk}(K_1, K_2, K_3) = \text{Tlk}(K_3, K_2, K_1) = 0$.*

Two n -component surface-links $F = K_1 \cup \dots \cup K_n$ and $F' = K'_1 \cup \dots \cup K'_n$ are *link homologous* if there is a compact oriented 3-manifold W properly embedded in $\mathbf{R}^4 \times [0, 1]$ such that W has n components W_1, \dots, W_n with $\partial W_i = K_i \times \{0\} \cup (-K'_i) \times \{1\}$. This relation is sometimes called *link-cobordism*, but that term also denotes the concordance relation. Since link homotopy implies link homology, the following theorem implies that triple linking invariants are link homotopy invariants (this fact is also seen from the sixth definition of Tlk). For related topics, refer to [3, 4, 6, 7, 8, 10, 13, 14].

Theorem 1.5 *Triple linking invariants are link homology invariants: If $F = K_1 \cup K_2 \cup K_3$ and $F' = K'_1 \cup K'_2 \cup K'_3$ are link homologous, then $\text{Tlk}(K_i, K_j, K_k) = \text{Tlk}(K'_i, K'_j, K'_k)$.*

See Remark 6.2 for further information about link homology.

By Theorem 1.2, for any 3-component surface-link $F = K_1 \cup K_2 \cup K_3$, there exists a pair of integers a and b such that

$$(*) \begin{cases} \text{Tlk}(K_1, K_2, K_3) = -\text{Tlk}(K_3, K_2, K_1) = -(a + b), \\ \text{Tlk}(K_2, K_3, K_1) = -\text{Tlk}(K_1, K_3, K_2) = b, \\ \text{Tlk}(K_3, K_1, K_2) = -\text{Tlk}(K_2, K_1, K_3) = a. \end{cases}$$

In [1], it is shown that for any pair of integers a and b , there exists a surface-link F whose triple linking numbers satisfy the above equations. However, that paper does not treat any problem about genera of the components of F . By Theorem 1.3, we see that

(1) if $a \neq 0$ and $b = 0$, then $g(K_i) \geq 1$ ($i = 1, 2$), and

(2) if $a \neq 0$, $b \neq 0$ and $a + b \neq 0$, then $g(K_i) \geq 1$ ($i = 1, 2, 3$),

where $g(K_i)$ denotes the genus of K_i .

Proposition 1.6 (i) *For any integer $a \neq 0$, there exists a surface-link $F = K_1 \cup K_2 \cup K_3$ whose triple linking numbers satisfy the above equations (*) with $b = 0$ and $g(K_i) = 1$ ($i = 1, 2$) and $g(K_3) = 0$.*

(ii) *For any pair of integers a and b with $a \neq 0$, $b \neq 0$ and $a + b \neq 0$, there exists a surface-link $F = K_1 \cup K_2 \cup K_3$ whose triple linking numbers satisfy the above equations (*) and $g(K_i) = 1$ ($i = 1, 2, 3$).*

This paper is organized as follows: in Section 2, we interpret Tlk_1 in terms of the decker curves of a surface diagram. In Section 3 we give precise definitions of triple linking numbers Tlk_i for $i = 2, \dots, 5$ (in terms of homology) and prove Theorem 1.1. Section 4 is devoted to proving Theorems 1.2–1.5. Proposition 1.6 is proved in Section 5.

Throughout this paper, all the homology and cohomology groups have the \mathbf{Z} -coefficient.

2 Decker Curves and Triple Linking

Let F be a surface-link and F^* a surface diagram of F with respect to a projection $p : \mathbf{R}^4 \rightarrow \mathbf{R}^3$. Let $\Gamma(F^*)$ denote the double point set of F^* ;

$$\{p(x) \mid x \in F, p(x) = p(y) \text{ for some } y \in F, x \neq y\},$$

which consists of immersed curves, called *double curves*. A double curve C^* is an immersed circle or an immersed arc in \mathbf{R}^3 . If C^* is an immersed circle, then $(p|_F)^{-1}(C^*) = C \cup C'$ for some pair of immersed circles C and C' in F . If C^* is an immersed arc, then its endpoints are branch points of F^* and $(p|_F)^{-1}(C^*) = C \cup C'$ for some pair of immersed arcs C and C' in F with $\partial C = \partial C'$. The curves C and C' are called *decker curves* over C^* : one of them is in higher position than the other with respect to the projection direction, which is called an *upper decker curve* and the other is called an *lower decker curve*. We notice that the preimage of a triple point consists of three points of F which are intersections of decker curves. See [2] for details.

Double curves and decker curves are oriented as follows: Let x be a point of F whose image $x^* = p(x)$ is not a branch point. There is a regular neighborhood N of x in F such that $p|_N$ is an embedding. An *orientation normal* \vec{n} to $N^* = p(N)$ in \mathbf{R}^3 at x^* is specified in such a way that $(\vec{v}_1, \vec{v}_2, \vec{n})$ matches the orientation of \mathbf{R}^3 , where the pair of tangents (\vec{v}_1, \vec{v}_2) defines the orientation of N^* that is induced from the orientation of $N \subset F$. If y is a double point on a double curve C^* , then C^* is locally an intersection of N_1^* and N_2^* , where N_1^* is upper and N_2^* is lower. We assign a tangent vector \vec{v} of C^* at y such that $(\vec{n}_1, \vec{n}_2, \vec{v})$ matches the orientation of \mathbf{R}^3 . This defines an orientation of C^* , cf. [1, 2]. We give an orientation to the lower decker curve over C^* such that it inherits the orientation from C^* , and give the opposite orientation to the upper decker curve. Note that, if C^* is an arc, then the orientations of C and C' are compatible (i.e., the union $C \cup C'$ forms an oriented immersed circle in F).

Let $F = K_1 \cup K_2 \cup K_3$ be a 3-component surface-link. A double curve C^* is *of type* (i, j) if the upper decker curve lies in K_i and the lower decker curve lies in K_j . A decker curve over C^* is *of type* (i, j) if C^* is so.

At a triple point τ , if the orientation normals to the top, middle, and bottom sheets at τ matches with this order the fixed orientation of \mathbf{R}^3 , then the sign of τ is positive and $\varepsilon(\tau) = +1$; otherwise the sign is negative and $\varepsilon(\tau) = -1$.

We interpret the triple linking Tlk_1 in terms of double decker curves as follows. Let D_{12}^ℓ (resp. D_{23}^u) denote the union of lower decker curves of type (1, 2) (resp. upper decker curves of type (2, 3)). Note that both D_{12}^ℓ and D_{23}^u are contained in K_2 .

Lemma 2.1 $\text{Tlk}_1(K_1, K_2, K_3) = -\text{Int}_{K_2}(D_{12}^\ell, D_{23}^u)$, where $\text{Int}_{K_2}(D_{12}^\ell, D_{23}^u)$ is the intersection number in K_2 .

Proof. Let τ be a triple point of type (1, 2, 3). The preimage of τ consists of three points of F . Exactly one of them is on K_2 and that is a double point of D_{12}^ℓ and D_{23}^u . Conversely the image of a double point of D_{12}^ℓ and D_{23}^u is a triple point of F^* of type (1, 2, 3). Hence there is a one-to-one correspondence between the set of triple points of type (1, 2, 3) and double points of D_{12}^ℓ and D_{23}^u . If the sign of τ is positive (or negative, resp.) then the corresponding intersection of D_{12}^ℓ and D_{23}^u is negative (resp. positive), see Figure 1. Thus we have the result. ■

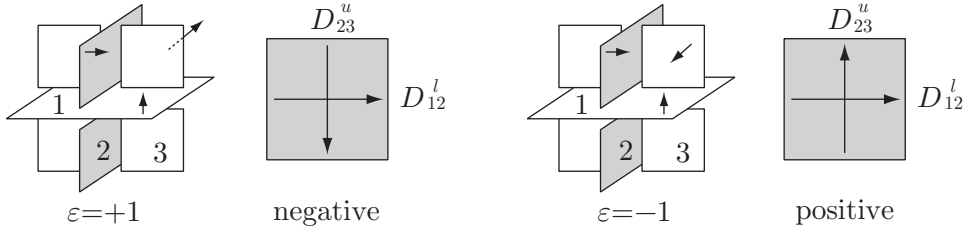


Figure 1: Triple points and intersection of decker curves

Since D_{12}^ℓ is the union of circles in \mathbf{R}^4 disjoint from K_3 , the linking number $\text{Link}(D_{12}^\ell, K_3)$ is defined.

Lemma 2.2 $\text{Tlk}_1(K_1, K_2, K_3) = \text{Link}(D_{12}^\ell, K_3)$.

Proof. Without loss of generality, we may assume that the projection p is given by $p(w, x, y, z) = (x, y, z)$. For a real number λ we denote by $t_\lambda : \mathbf{R}^4 \rightarrow \mathbf{R}^4$ the translation with $t_\lambda(w, x, y, z) = (w + \lambda, x, y, z)$. Let M'_3 be a 3-chain in \mathbf{R}^4 with $\partial M'_3 = K_3$. Take a sufficiently large number R and consider a 3-chain

$$M_3 = \cup_{\lambda \in [0, R]} t_\lambda(K_3) + t_R(M'_3)$$

so that $\partial M_3 = K_3$ and $D_{12}^\ell \cap M_3 = D_{12}^\ell \cap (\cup_{\lambda \in [0, R]} t_\lambda(K_3))$. The projection p induces a one-to-one correspondence between the geometric intersection $D_{12}^\ell \cap (\cup_{\lambda \in [0, R]} t_\lambda(K_3))$ and the subset of $D_{12}^{\ell*} \cap K_3^* = p(D_{12}^\ell) \cap p(K_3)$ consisting of points where $D_{12}^{\ell*}$ is higher than K_3^* (in the over-under information of the surface diagram F^*), i.e., the set of triple points of F^* of type (1, 2, 3). Since the orientation of D_{12}^ℓ is parallel to the orientation of $D_{12}^{\ell*}$, the sign of an intersection of D_{12}^ℓ and $\cup_{\lambda \in [0, R]} t_\lambda(K_3)$ coincides with the sign of the corresponding intersection of $D_{12}^{\ell*}$ and K_3^* , which is the sign of the triple point (see Figure 2). Thus we have the result. ■

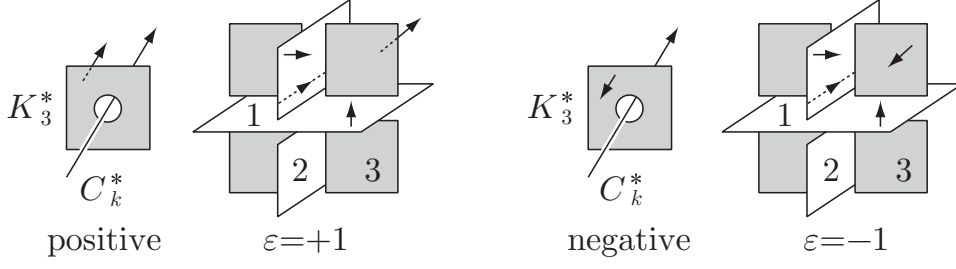


Figure 2: The intersection between K_3 and D_{12}^ℓ

3 Proof of Theorem 1.1

For a compact oriented n -manifold M with $\{A, B\} = \{\partial M, \emptyset\}$, we denote by

$$\cdot_M : H_p(M, A) \times H_q(M, A) \rightarrow H_{p+q-n}(M, A)$$

the intersection map, which is defined by

$$x \cdot_M y = P_M(P_M^{-1}(x) \cup P_M^{-1}(y))$$

where $P_M : H^*(M, B) \rightarrow H_{n-*}(M, A)$ is the Poincaré duality isomorphism (see [9], page 391). We will use \cdot and P instead of \cdot_M and P_M when their meanings are obvious in context.

Let $F = K_1 \cup K_2 \cup K_3$ be a 3-component surface-link. For simplicity of argument, we assume that F is embedded in the 4-sphere $S^4 = \mathbf{R}^4 \cup \{\infty\}$. For a regular neighborhood N_i of K_i in S^4 , we put

$$E_i = \text{Cl}(S^4 \setminus N_i), \quad E_{ij} = \text{Cl}(S^4 \setminus (N_i \cup N_j)) \text{ for } i \neq j, \text{ and } E = \text{Cl}(S^4 \setminus (N_1 \cup N_2 \cup N_3)),$$

where Cl denotes the closure. We denote by M_i a 3-chain in S^4 with $\partial M_i = K_i$ for $i = 1, 2, 3$ (the reader may suppose that it is a Seifert hypersurface for K_i , i.e., a compact oriented 3-manifold embedded in S^4 with $\partial M_i = K_i$). We also denote by M_i the homology class in

$$H_3(S^4, K_i) \cong H_3(S^4, N_i) \cong H_3(E_i, \partial E_i)$$

represented by M_i . By $u_i \in H^1(E_i)$ we denote the Poincaré dual of $M_i \in H_3(E_i, \partial E_i)$, i.e., $M_i = P(u_i) = u_i \cap [E_i]$. For a subset X of E_i , we will denote by $u_i|_X \in H^1(X)$ the image of u_i by the inclusion-induced homomorphism $H^1(E_i) \rightarrow H^1(X)$. Moreover, if X is an n -manifold, we denote by $M_i|_{(X, \partial X)}$ (or $M_i|_X$ if $\partial X = \emptyset$) the Poincaré dual $P_X(u_i|_X) = (u_i|_X) \cap [X] \in H_{n-1}(X, \partial X)$ of $u_i|_X$.

For $i \in \{1, 3\}$, since $K_2 \subset E_i$, $M_i|_{K_2} \in H_1(K_2)$ is defined. (When we consider M_i as a 3-chain, the intersection of M_i and K_2 (as a 1-cycle in K_2) represents $M_i|_{K_2}$.) Let

$$\text{Tk}_2(K_1, K_2, K_3) = \varepsilon_{K_2}(M_1|_{K_2} \cdot M_3|_{K_2}),$$

where $\varepsilon_{K_2} : H_0(K_2) \rightarrow \mathbf{Z}$ is the augmentation.

Lemma 3.1 $\text{Tlk}_1(K_1, K_2, K_3) = \text{Tlk}_2(K_1, K_2, K_3)$.

Proof. We assume $F \subset \mathbf{R}^4 \subset S^4$ and continue the situation of the proof of Lemma 2.2. Let M'_1 be a 3-chain in $\mathbf{R}^4 \subset S^4$ with $\partial M'_1 = K_1$ and consider a 3-chain M_1 such that

$$M_1 = -\cup_{\lambda \in [-R, 0]} t_\lambda(K_1) + t_{-R}(M'_1)$$

with $\partial M_1 = K_1$. The intersection of M_1 and K_2 is equal to that of $-\cup_{\lambda \in [-R, 0]} t_\lambda(K_1)$ and K_2 which is the 1-chain $-D_{12}^\ell$ in K_2 , and the intersection of M_3 and K_2 is equal to that of $\cup_{\lambda \in [0, R]} t_\lambda(K_3)$ and K_2 which is the 1-chain D_{23}^u in K_2 . Therefore, by Lemma 2.1, we have

$$\begin{aligned} \text{Tlk}_1(K_1, K_2, K_3) &= -\text{Int}_{K_2}(D_{12}^\ell, D_{23}^u) \\ &= -\varepsilon_{K_2}(D_{12}^\ell \cdot D_{23}^u) \\ &= -\varepsilon_{K_2}(-M_1|_{K_2} \cdot M_3|_{K_2}) \\ &= \text{Tlk}_2(K_1, K_2, K_3). \blacksquare \end{aligned}$$

Remark. The argument in Lemmas 2.2 and 3.1 implies that for any Seifert hypersurface M_1 for K_1 , the intersection of M_1 and K_2 (as a 1-cycle in K_2) is homologous to $-D_{12}^\ell$, and that for any Seifert hypersurface M_3 for K_3 , the intersection of M_3 and K_2 (as a 1-cycle in K_2) is homologous to D_{23}^u .

We denote by $[M_1 \cap K_2]_{E_3} \in H_1(E_3)$ the homology class of the intersection $M_1 \cap K_2$ as a 1-cycle in E_3 . This is equal to the image of $M_1|_{K_2} \in H_1(K_2)$ under the inclusion-induced homomorphism $H_1(K_2) \rightarrow H_1(E_3)$ and also equal to the image of $M_1|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}} \in H_1(E_{13})$ under the inclusion-induced homomorphism $H_1(E_{13}) \rightarrow H_1(E_3)$, where $[K_2]_{E_{13}} \in H_2(E_{13})$ is represented by K_2 . Let

$$\begin{aligned} \text{Tlk}_3(K_1, K_2, K_3) &= \text{Link}([M_1 \cap K_2]_{E_3}, K_3) \\ &= \varepsilon_{E_3}([M_1 \cap K_2]_{E_3} \cdot M_3), \end{aligned}$$

where $M_3 \in H_3(E_3, \partial E_3)$ and $\varepsilon_{E_3} : H_0(E_3) \rightarrow \mathbf{Z}$ is the augmentation.

Lemma 3.2 $\text{Tlk}_1(K_1, K_2, K_3) = -\text{Tlk}_3(K_1, K_2, K_3)$.

Proof. In the situation of the proof of Lemma 2.2, $[M_1 \cap K_2]_{E_3} \in H_1(E_3)$ is represented by the 1-cycle $-D_{12}^\ell$. Therefore, by Lemma 2.2, we have

$$\begin{aligned} \text{Tlk}_3(K_1, K_2, K_3) &= \text{Link}([M_1 \cap K_2]_{E_3}, K_3) \\ &= \text{Link}(-D_{12}^\ell, K_3) \\ &= -\text{Tlk}_1(K_1, K_2, K_3). \blacksquare \end{aligned}$$

We denote by $[M_1 \cap M_3]_{(E_{13}, \partial E_{13})} \in H_1(E_{13}, \partial E_{13})$ the class of the intersection $M_1 \cap M_3$ as a 2-cycle in $(E_{13}, \partial E_{13})$ when we regard M_i as a 3-chain. This is equal to the intersection product $M_1|_{(E_{13}, \partial E_{13})} \cdot M_3|_{(E_{13}, \partial E_{13})} \in H_2(E_{13}, \partial E_{13})$. Let

$$\begin{aligned} \text{Tlk}_4(K_1, K_2, K_3) &= \varepsilon_{E_{13}}([M_1 \cap M_3]_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}}) \\ &= \varepsilon_{E_{13}}(M_1|_{(E_{13}, \partial E_{13})} \cdot M_3|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}}). \end{aligned}$$

Lemma 3.3 $\text{Tlk}_3(K_1, K_2, K_3) = \text{Tlk}_4(K_1, K_2, K_3)$.

Proof. Let $i_* : H_*(E_{13}) \rightarrow H_*(E_3)$ and $i^* : H^*(E_3) \rightarrow H^*(E_{13})$ be the inclusion-induced homomorphisms. Recall that $[M_1 \cap K_2]_{E_3} = i_*(M_1|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}})$. Thus,

$$\begin{aligned}
\text{Tlk}_4(K_1, K_2, K_3) &= \varepsilon_{E_{13}}(M_1|_{(E_{13}, \partial E_{13})} \cdot M_3|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}}) \\
&= \varepsilon_{E_3} \circ i_*(M_1|_{(E_{13}, \partial E_{13})} \cdot M_3|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}}) \\
&= -\varepsilon_{E_3} \circ i_*(M_3|_{(E_{13}, \partial E_{13})} \cdot M_1|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}}) \\
&= -\varepsilon_{E_3} \circ i_*(u_3|_{E_{13}} \cap (M_1|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}})) \\
&= -\varepsilon_{E_3} \circ i_*(i^*(u_3) \cap (M_1|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}})) \\
&= -\varepsilon_{E_3}(u_3 \cap i_*(M_1|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}})) \\
&= -\varepsilon_{E_3}(M_3 \cdot [M_1 \cap K_2]_{E_3}) \\
&= \varepsilon_{E_3}([M_1 \cap K_2]_{E_3} \cdot M_3) \\
&= \text{Tlk}_3(K_1, K_2, K_3). \blacksquare
\end{aligned}$$

For $i \in \{1, 2, 3\}$, since $\partial N_2 \subset E_i$, $M_i|_{\partial N_2} \in H_2(\partial N_2)$ is defined. Let

$$\begin{aligned}
\text{Tlk}_5(K_1, K_2, K_3) &= \varepsilon_{\partial N_2}(M_1|_{\partial N_2} \cdot M_2|_{\partial N_2} \cdot M_3|_{\partial N_2}) \\
&= \langle u_1|_{\partial N_2} \cup u_2|_{\partial N_2} \cup u_3|_{\partial N_2}, [\partial N_2] \rangle.
\end{aligned}$$

Lemma 3.4 $\text{Tlk}_4(K_1, K_2, K_3) = \text{Tlk}_5(K_1, K_2, K_3)$.

Proof. Let $i : \partial N_2 \rightarrow N_2$ be the inclusion map. In $H_0(N_2)$, we have

$$\begin{aligned}
i_*(M_1|_{\partial N_2} \cdot M_2|_{\partial N_2} \cdot M_3|_{\partial N_2}) &= -i_*(M_1|_{\partial N_2} \cdot M_3|_{\partial N_2} \cdot M_2|_{\partial N_2}) \\
&= -i_*(\partial_*(M_1|_{(N_2, \partial N_2)}) \cdot \partial_*(M_3|_{(N_2, \partial N_2)}) \cdot M_2|_{\partial N_2}) \\
&= -i_*(\partial_*(M_1|_{(N_2, \partial N_2)}) \cdot M_3|_{(N_2, \partial N_2)}) \cdot M_2|_{\partial N_2} \\
&= -(M_1|_{(N_2, \partial N_2)} \cdot M_3|_{(N_2, \partial N_2)}) \cdot i_*(M_2|_{\partial N_2}) \\
&= -M_1|_{(N_2, \partial N_2)} \cdot M_3|_{(N_2, \partial N_2)} \cdot (-[K_2]_{N_2}) \\
&= M_1|_{(N_2, \partial N_2)} \cdot M_3|_{(N_2, \partial N_2)} \cdot [K_2]_{N_2}.
\end{aligned}$$

Thus

$$\text{Tlk}_5(K_1, K_2, K_3) = \varepsilon_{N_2}(M_1|_{(N_2, \partial N_2)} \cdot M_3|_{(N_2, \partial N_2)} \cdot [K_2]_{N_2}).$$

It is obvious that

$$\varepsilon_{N_2}(M_1|_{(N_2, \partial N_2)} \cdot M_3|_{(N_2, \partial N_2)} \cdot [K_2]_{N_2}) = \varepsilon_{E_{13}}(M_1|_{(E_{13}, \partial E_{13})} \cdot M_3|_{(E_{13}, \partial E_{13})} \cdot [K_2]_{E_{13}})$$

and hence we have the result. \blacksquare

Lemma 3.5 $\text{Tlk}_6(K_1, K_2, K_3) = \pm \text{Tlk}_1(K_1, K_2, K_3)$.

Proof. Since Tlk_6 is an ambient isotopy invariant, we may assume that the surface-link $F = f(F_1) \cup f(F_2) \cup f(F_3)$ is in general position with respect to the projection $p : \mathbf{R}^4 \rightarrow \mathbf{R}^3$ with $p(w, x, y, z) = (x, y, z)$. The preimage of a particular point $((1, 0, 0, 0), (1, 0, 0, 0))$ by L consists of triples $(x_1, x_2, x_3) \in F_1 \times F_2 \times F_3$ such that $p(f(x_1)) = p(f(x_2)) = p(f(x_3))$ and

$f(x_1)$ is the upper, $f(x_2)$ is the middle, $f(x_3)$ is the lower lift of the triple point $p(f(x_1))$. For each such triple (x_1, x_2, x_3) , let D_T^2 , D_M^2 , D_B^2 be regular neighborhoods of them in $F_1 \cup F_2 \cup F_3$, and let $\varepsilon \in \{+1, -1\}$ be the sign of the triple point $p(f(x_1))$. Let (x_T, y_T) , $(x_M, -\varepsilon z_M)$ and (y_B, z_B) be coordinate systems of D_T^2 , D_M^2 and D_B^2 around x_1, x_2 and x_3 , respectively. Modifying f up to ambient isotopy, we may assume that the restriction of f to $D_T^2 \cup D_M^2 \cup D_B^2$ is given by defined by

$$\begin{aligned} (x_T, y_T) &\mapsto (0, x_0, y_0, z_0) + (3, x_T, y_T, 0) \\ (x_M, -\varepsilon z_M) &\mapsto (0, x_0, y_0, z_0) + (2, x_M, 0, z_M) \\ (y_B, z_B) &\mapsto (0, x_0, y_0, z_0) + (1, y_B, z_B) \end{aligned}$$

where $(x_0, y_0, z_0) \in \mathbf{R}^3$ is the triple point $p(f(x_1))$. In this situation, the restriction

$$L' : D_T^2 \times D_M^2 \times D_B^2 \rightarrow S^3 \times S^3$$

is given by the formula

$$\left(\frac{(1, x_T - x_M, y_T, -z_M)}{\sqrt{1 + (x_T - x_M)^2 + y_T^2 + z_M^2}}, \frac{(1, x_M, -y_B, z_M - z_B)}{\sqrt{1 + x_M^2 + y_B^2 + (z_M - z_B)^2}} \right).$$

The map L' is injective and hence it is a homeomorphism onto its image. Its (local) degree is $+1$ or -1 which depends only on ε . Since the degree of L is the sum of the (local) degrees of L' for all triples (x_1, x_2, x_3) in the preimage $L^{-1}((1, 0, 0, 0), (1, 0, 0, 0))$, this number agrees up to sign with the triple linking number $\text{Trk}_1(f(F_1), f(F_2), f(F_3))$. ■

By Lemmas 3.1–3.5, we have Theorem 1.1.

4 Proof of Theorems 1.2–1.5

To prove Theorem 1.2, it is useful to change ∂N_2 in the definition of Trk_5 for ∂E_2 .

Lemma 4.1 $\text{Trk}_5(K_1, K_2, K_3) = -\varepsilon_{\partial E_2}(M_1|_{\partial E_2} \cdot M_2|_{\partial E_2} \cdot M_3|_{\partial E_2})$, where the intersections are taken in ∂E_2 .

Proof. Since ∂N_2 and ∂E_2 are the same 3-submanifold of S^4 with opposite orientations, $[\partial N_2] = -[\partial E_2]$ in $H_3(\partial N_2) = H_3(\partial E_2)$. Thus, in $H_0(\partial N_2) = H_0(\partial E_2)$,

$$\begin{aligned} M_1|_{\partial N_2} \cdot M_2|_{\partial N_2} \cdot M_3|_{\partial N_2} &= (u_1|_{\partial N_2} \cup u_2|_{\partial N_2} \cup u_3|_{\partial N_2}) \cup [\partial N_2] \\ &= (u_1|_{\partial E_2} \cup u_2|_{\partial E_2} \cup u_3|_{\partial E_2}) \cup (-[\partial E_2]) \\ &= -M_1|_{\partial E_2} \cdot M_2|_{\partial E_2} \cdot M_3|_{\partial E_2}. \blacksquare \end{aligned}$$

Proof of Theorem 1.2 (i)

$$\begin{aligned} \text{Trk}_2(K_1, K_2, K_3) &= \varepsilon_{K_2}(M_1|_{K_2} \cdot M_3|_{K_2}) \\ &= -\varepsilon_{K_2}(M_3|_{K_2} \cdot M_1|_{K_2}) \\ &= -\text{Trk}_2(K_3, K_2, K_1). \end{aligned}$$

(ii) Note that $M_i|_{(E, \partial E)} = P_E(u_i|_E) \in H_3(E, \partial E)$ is the image of M_i under

$$H_3(E_i, \partial E_i) \rightarrow H_3(E_i, \partial E_i \cup N_j \cup N_k) \cong H_3(E, \partial E),$$

and $M_i|_{\partial E_2} \in H_2(\partial E_2)$ is the image of $M_i|_{(E, \partial E)}$ under

$$H_3(E, \partial E) \rightarrow H_2(\partial E) \cong H_2(\partial E_1) \oplus H_2(\partial E_2) \oplus H_2(\partial E_3) \rightarrow H_2(\partial E_2),$$

the boundary operator followed by the projection to $H_2(\partial E_2)$. We denote by $(M_i|_{\partial E_2})_{\partial E} \in H_2(\partial E)$ the image of $M_i|_{\partial E_2}$ under the inclusion-induced homomorphism $H_2(\partial E_2) \rightarrow H_2(\partial E)$. By Lemma 4.1,

$$\text{Tlk}_5(K_1, K_2, K_3) = -\varepsilon_{\partial E}((M_1|_{\partial E_2})_{\partial E} \cdot (M_2|_{\partial E_2})_{\partial E} \cdot (M_3|_{\partial E_2})_{\partial E}).$$

Thus, we have

$$\begin{aligned} \text{Tlk}_5(K_1, K_2, K_3) + & \text{Tlk}_5(K_2, K_3, K_1) + \text{Tlk}_5(K_3, K_1, K_2) \\ = & -\varepsilon_{\partial E}((M_1|_{\partial E_2})_{\partial E} \cdot (M_2|_{\partial E_2})_{\partial E} \cdot (M_3|_{\partial E_2})_{\partial E} \\ & + (M_2|_{\partial E_3})_{\partial E} \cdot (M_3|_{\partial E_3})_{\partial E} \cdot (M_1|_{\partial E_3})_{\partial E} \\ & + (M_3|_{\partial E_1})_{\partial E} \cdot (M_1|_{\partial E_1})_{\partial E} \cdot (M_2|_{\partial E_1})_{\partial E}) \\ = & -\varepsilon_{\partial E}((M_1|_{\partial E_2})_{\partial E} \cdot (M_2|_{\partial E_2})_{\partial E} \cdot (M_3|_{\partial E_2})_{\partial E} \\ & + (M_1|_{\partial E_3})_{\partial E} \cdot (M_2|_{\partial E_3})_{\partial E} \cdot (M_3|_{\partial E_3})_{\partial E} \\ & + (M_1|_{\partial E_1})_{\partial E} \cdot (M_2|_{\partial E_1})_{\partial E} \cdot (M_3|_{\partial E_1})_{\partial E}) \\ = & -\varepsilon_{\partial E}(M_1|_{\partial E} \cdot M_2|_{\partial E} \cdot M_3|_{\partial E}) \\ = & -\varepsilon_{\partial E}(\partial_*(M_1|_{(E, \partial E)}) \cdot \partial_*(M_2|_{(E, \partial E)}) \cdot \partial_*(M_3|_{(E, \partial E)})) \\ = & -\varepsilon_{\partial E}(\partial_*(M_1|_{(E, \partial E)}) \cdot M_2|_{(E, \partial E)} \cdot M_3|_{(E, \partial E)}) \\ = & 0. \blacksquare \end{aligned}$$

Proof of Theorem 1.3 (i) The intersection number between two oriented curves on a 2-sphere vanishes. By Lemma 2.1, we have $\text{Tlk}(K_1, K_2, K_3) = 0$.

(ii) This is an immediate consequence of (i) and Theorem 1.2(ii). \blacksquare

Proof of Theorem 1.4 If $\text{Alk}(K_2, K_3) = 0$, then $\text{Tlk}_3(K_1, K_2, K_3) = \text{Link}([M_1 \cap K_2]_{E_3}, K_3) = 0$, for $[M_1 \cap K_2]_{E_3} \in H_1(E_3)$ is the image of $M_1|_{K_2} \in H_1(K_2) = 0$. By Theorem 1.2, we have $\text{Tlk}(K_3, K_2, K_1) = 0$. \blacksquare

We consider surface-links $F = K_1 \cup K_2 \cup K_3$ in which each K_i is not necessarily connected. Such a surface-link is called a 3-partitioned surface-link. The definition of the triple linking of $F = K_1 \cup K_2 \cup K_3$ is generalized directly for 3-partitioned surface-links, and all results and proofs in Sections 2 and 3 are valid for 3-partitioned surface-links. Theorem 1.5 is a special case of the following:

Theorem 4.2 *If two 3-partitioned surface-links $F = K_1 \cup K_2 \cup K_3$ and $F' = K'_1 \cup K'_2 \cup K'_3$ are link homologous, then $\text{Tlk}(K_i, K_j, K_k) = \text{Tlk}(K'_i, K'_j, K'_k)$.*

Proof. It is sufficient to prove $\text{Tlk}(K_1, K_2, K_3) = \text{Tlk}(K'_1, K'_2, K'_3)$ in a special case that $K_i = K'_i$, $K_j = K'_j$ and K_k is homologous to K'_k in $S^4 \setminus (K_i \cup K_j)$, where $\{i, j, k\} =$

$\{1, 2, 3\}$. If $k = 2$, then $\text{Tlk}_4(K_1, K_2, K_3) = \text{Tlk}_4(K_1, K'_2, K_3)$ by definition. If $k = 1$, then $\text{Tlk}_3(K_1, K_2, K_3) = \text{Tlk}_3(K'_1, K_2, K_3)$. (This is seen as follows: Let M_1 be a 3-chain with $\partial M_1 = K_1$. Since K'_1 is homologous to K_1 in $S^4 \setminus (K_2 \cup K_3)$, there is a 3-chain B in $S^4 \setminus (K_2 \cup K_3)$ with $\partial B = K'_1 - K_1$. Let $M'_1 = M_1 + B$, which is a 3-chain with $\partial M'_1 = K'_1$. Then $[M_1 \cap K_2]_{E_3} = [M'_1 \cap K_2]_{E_3}$ in $H_1(E_3) \cong H_1(S^4 \setminus K_3)$.) The case $k = 3$ is reduced to the previous case ($k = 1$) by use of Theorem 1.2(i). ■

5 Proof of Proposition 1.6

(1) Let $\ell = k_1 \cup k_2$ be a $(2, 2a)$ -torus link in a 3-disk D^3 with $\text{Link}(k_1, k_2) = a$. Let γ be a simple loop in \mathbf{R}^4 which intersects a 3-disk B_0 in \mathbf{R}^4 transversely at a single interior point of B_0 in the positive direction. Identify $D^3 \times S^1$ with a regular neighborhood $N(\gamma)$ of γ in \mathbf{R}^4 and let $T_1 \cup T_2$ be the image of $\ell \times S^1 = k_1 \times S^1 \cup k_2 \times S^1$ in \mathbf{R}^4 . Let $F = K_1 \cup K_2 \cup K_3$ be a surface-link with $K_1 = T_1$, $K_2 = T_2$ and $K_3 = \partial B_0$. Then F is the desired link.

(2) Let $\ell = k_1 \cup k_2 \cup k_3$ be a pretzel link of type $(2a, -2b)$ in a 3-disk D^3 so that $\text{Link}(k_1, k_2) = a$, $\text{Link}(k_2, k_3) = -b$ and $\text{Link}(k_1, k_3) = 0$. Let B_1, B_2, B_3 be mutually disjoint 3-disks embedded in \mathbf{R}^4 and let γ be a simple loop in \mathbf{R}^4 which intersects B_i ($i = 1, 2, 3$) transversely at a single interior point of B_i in the positive direction. Identify $D^3 \times S^1$ with a regular neighborhood $N(\gamma)$ of γ in \mathbf{R}^4 and let $T_1 \cup T_2 \cup T_3$ be the image of $\ell \times S^1 = k_1 \times S^1 \cup k_2 \times S^1 \cup k_3 \times S^1$ in \mathbf{R}^4 . Let $F = K_1 \cup K_2 \cup K_3$ be a surface-link obtained from $(T_1 \cup T_2 \cup T_3) \cup (\partial B_1 \cup \partial B_2 \cup \partial B_3)$ by piping such that F has a projection as in Figure 3. Then F is the desired link. ■

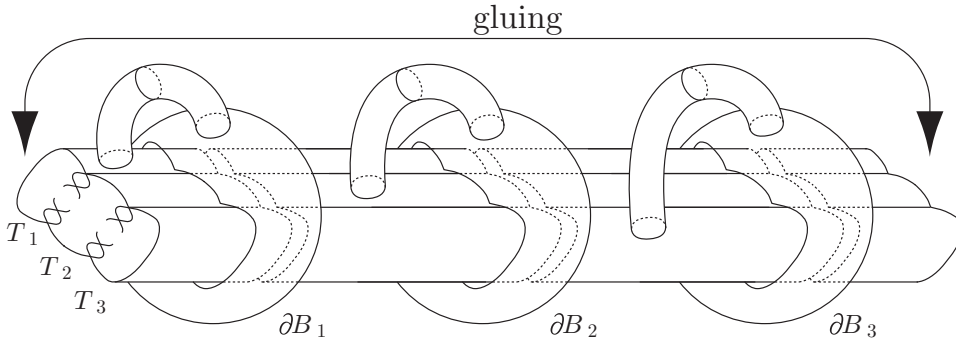


Figure 3: Linked tori with given linking invariants

6 Remarks

Remark 6.1 The definition of Tlk_6 can be seen as a direct analogue of (6) given in [11] page 133. See also [5, 10]. This generalizes the triple linking to all link maps, instead of embeddings. Moreover, it easily generalized to all dimensions. Let M_i denote a closed connected n -manifold for $i = 1, \dots, n+1$. Let an embedding $f : \cup_{i=1}^{n+1} M_i \rightarrow \mathbf{R}^{n+2}$ be given. Define $L : \prod_{i=1}^{n+1} M_i \rightarrow \prod_{j=1}^n S_j^{n+1}$ as follows. Let $x_i \in M_i$; for $i = 1, \dots, n$, let

$\Delta_i = f(x_i) - f(x_{i+1}) / \|f(x_i) - f(x_{i+1})\|$. Then

$$L(x_1, \dots, x_{n+1}) = (\Delta_1, \Delta_2, \dots, \Delta_n).$$

The general $(n+1)$ -fold linking number, Glk , is defined by

$$\text{Glk}(f(M_1), \dots, f(M_{n+1})) = \deg(L).$$

We can generalize the notion of Tlk_1 to a diagram in \mathbf{R}^{n+1} of an $(n+1)$ -component n -manifold-link $M_1 \cup \dots \cup M_{n+1}$ in \mathbf{R}^{n+2} ; namely, a diagram has generic $(n+1)$ -tuple points and we count the number of times M_1 is over M_2 is over ... is over M_{n+1} with signs. It is difficult to show that this value is an invariant of the n -manifold-link in \mathbf{R}^{n+2} directly, since we do not know Reidemeister moves for higher dimensions ($n \geq 3$). However, the proof of Lemma 3.5 goes through to show that Glk is the same as this count (up to sign). Thus we have that this number (generalization of Tlk_1) is an invariant of an $(n+1)$ -component n -manifold-link.

Remark 6.2 In classical link theory, the linking number determines the link homology classes completely. However, the triple linking of surface-links is *not* a complete invariant of the surface-link homology; there exists a pair of surface-links with the same triple linking invariants which are not link homologous. A classification of surface-link homology classes is discussed in a forthcoming paper.

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